Minkowski Spacetime Diagrams

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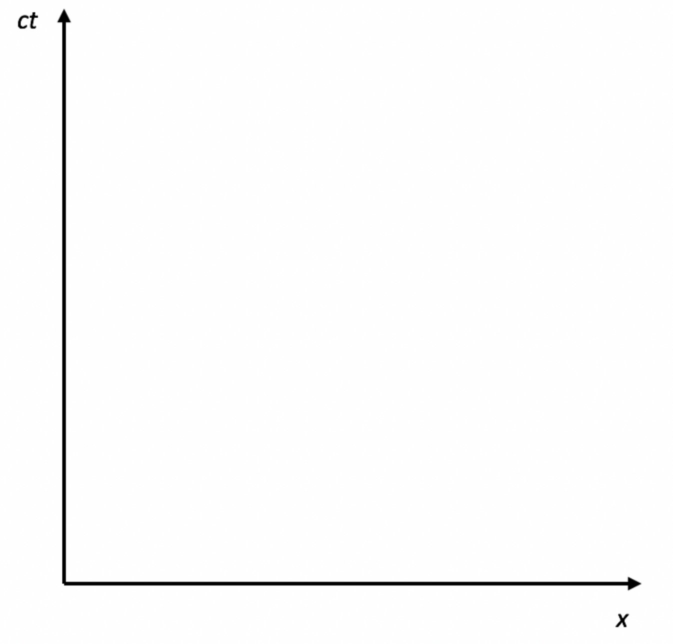
Abstract

This paper offers an overview of the way relativistic events are represented through the construction of Minkowski spacetime diagrams, graphical structures that relate the spatial and temporal features of objects as they move at an appreciable fraction of the speed of light. The postulates that provide the basis for Albert Einstein's famous theory of special relativity will be defined. With these guiding principles in mind, we will attempt to make sense of a number of paradoxical results that contradict the findings of classical Newtonian physics. After discussing the nature of multiple reference frames and deriving the transformations that enable mathematical translation between them, we will prove that length and time vary under special circumstances and solve a classic Einsteinian problem deconstructing the illusion of simultaneity.

Introduction

Spacetime diagrams were introduced in 1908 by Hermann Minkowski to visualize the properties of space and time under relativistic motion in different frames of reference. The basic diagram consists of perpendicular axes akin to those found in a Cartesian plane. Instead of dimensional coordinates, however, spacetime diagrams plot space (x) along the horizontal axis and time (t) along the vertical. Points in spacetime diagrams, given by (x, t), are known as 'events', and the lines traced by movement through space and time are called 'worldlines'.

The time axis is often labelled as ct instead of simply t, where c represents the universal constant for the speed of light. The addition of the constant is simply a convention of units that will, as we shall see, allow us to plot the speed of light as a line 45° from the horizontal axis. We could alternatively elect to use units where $c = 1$.



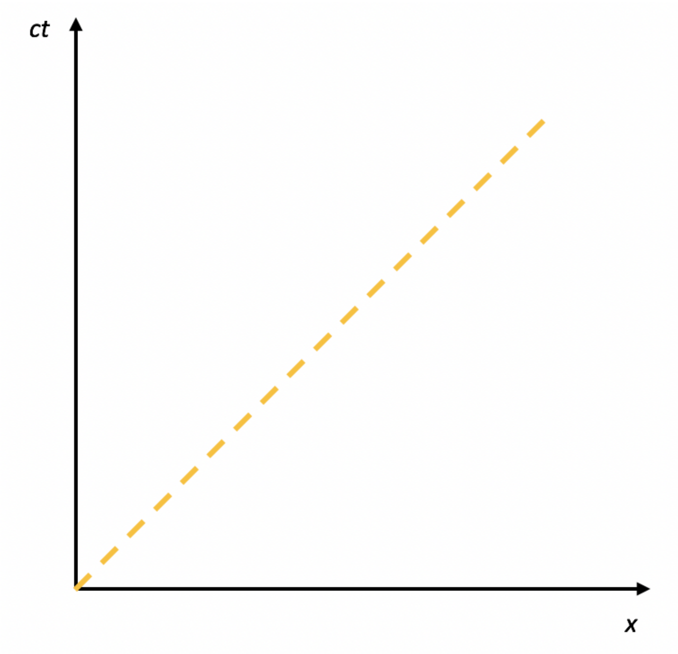
Fundamental Postulates of Special Relativity

Two postulates guide any approach to analyzing relativistic movement and provide the basis for understanding and constructing spacetime diagrams.

The first of these postulates claims that all inertial references frames are equivalent. A reference frame is the point of view from which an observer experiences the physics of the surrounding environment, and an inertial reference frame is one that does not accelerate. Inertial reference frames include frames which are stationary or which move at constant velocity. The first postulate maintains that no inertial reference frame is absolute or preferred over another. No frame is inherently 'moving', for all movement is relative to some resting frame. It also implies that the laws of physics are the same across all inertial reference frames, an assertion that recalls the ubiquitous train often cited in problems concerning relativity: if one conducts an experiment within a sealed train car traveling at a constant speed and isolated from stimuli from the outside world, one would have no way of discerning fact of the train car's movement, for the experimental results would be identical to those obtained in a stationary setting.

The second postulate claims that the speed of light is not only the cosmic speed limit, but that it is consistent across any reference frame. This fact has been confirmed through repeated experimental tests, though it seems to violate our natural intuitions about relative speeds. The speeds we observe, however, are nowhere close to relativistic proportions, which is why relativistic effects are not pronounced and can thus be ignored in everyday life.

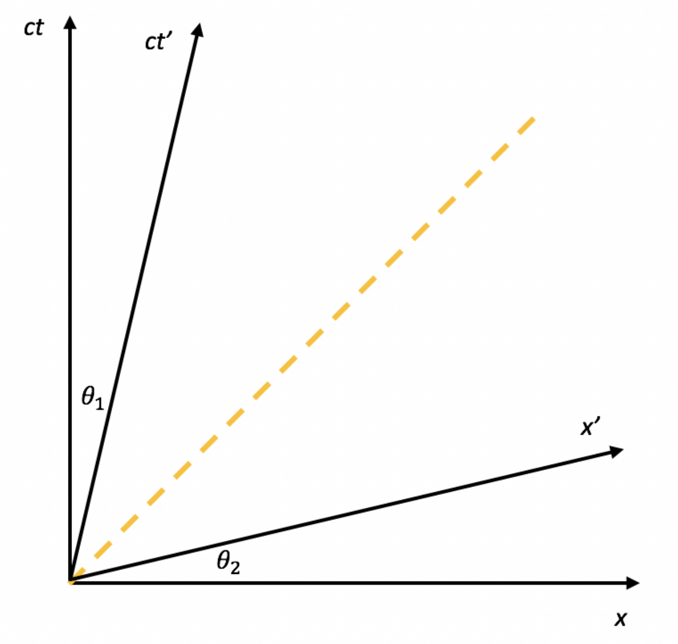
This second postulate has important implications for spacetime diagrams. The speed of light, c = 3.8 x 10^8 m/s, never varies, meaning its worldline remains at a fixed slope no matter the reference frame or spacetime scenario. We can now see why it makes sense to plot ct on the vertical spacetime axis in order to ensure that light beams travel at a 45° angle, representative of one meter of spatial translation per one second.



No worldline representing a moving object can be drawn with a slope greater than c, since this would imply that the object traveled a greater spatial distance in a shorter timespan than light, which we know to be impossible. The light beam as drawn above separates 'timelike' and 'spacelike' events. Timelike events are those between which it is possible for information to travel. One can causally affect the other, and there is some reference frame where the events happen at the same place, but it is impossible for them to occur simultaneously. Spacelike events, meanwhile, can be simultaneous in some reference frame, but there is no possible way for information to travel between them quickly enough to causally affect one another.

Lorentz Transformations

The purpose of any spacetime diagram is to illustrate the spatial and temporal relations between two frames of reference, one moving relative to the other, and to locate events as observed in space and time from the perspective of each frame. For a given diagram, the resting frame will be specified by the usual, perpendicular space and time axes, but we must construct the other, moving frame, which will be represented by axes oriented at an angle to denote their velocity from the perspective of the resting frame. Conventional notation differentiates between the two sets of axes with primes, as shown below, where the moving frame is primed and the rest frame unprimed.



Since we seek a way to relate the coordinates of a single event in each reference frame, we must utilize equations that express x and t coordinates in terms of their x' and t' analogues. These equations are known as the Lorentz transformations,

$$x = \gamma(x' + vt')$$
$$t = \gamma(t' + vx')$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Rearranging to solve for x' and t', we can also obtain the inverse Lorentz equations:

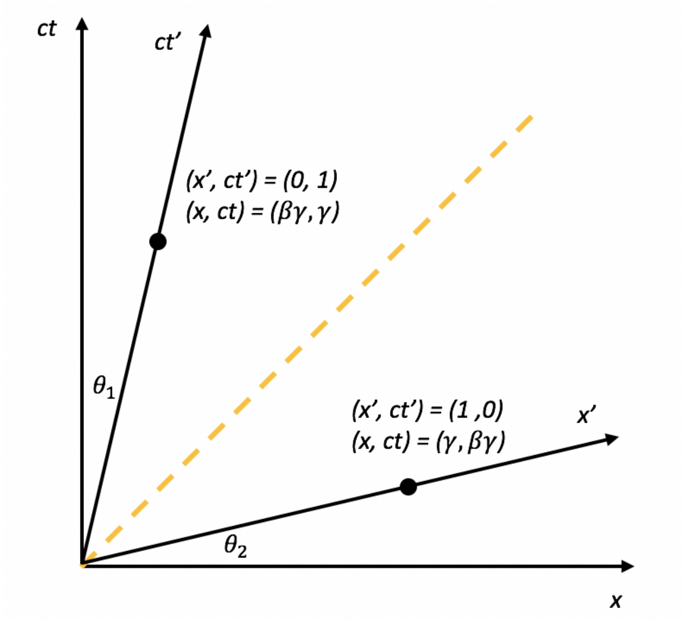
$$x' = \gamma(x - vt)$$
$$t' = \gamma(x + vt)$$

Returning to the spacetime diagram, we can use the Lorentz transformations to plot points on the primed axes and determine their locations in the resting, unprimed frame. The fraction $\frac{v}{c}$ is often expressed as β.

Knowing the spatial and temporal coordinates of each point, simple geometry allows us to measure the two angles that always define the tilt of the moving frame compared to the frame at rest.

$$\tan\theta_1 = \frac{x}{ct} = \beta$$

$$\tan\theta_2 = \frac{ct}{x} = \beta$$



The x' and ct' axes are both tilted by the same angle, and it can be shown through calculation of the distance from each point to the origin that their ratios of primed units to unprimed units are also equal, at a ratio of

$$\sqrt{\frac{1+\beta^2}{1-\beta^2}}$$

It is worth noting that simultaneous events are found along lines parallel to the spatial axis of a given reference frame, since points along those lines all share the same temporal coordinate. Simultaneous events in the unprimed frame would lie along horizontal lines parallel to the x axis, while those in the primed frame would fall along lines parallel to the x' axis, offset from the horizontal by the angle $\theta_2 = \tan^{-1}\beta$.

Loss of Simultaneity

The construction of spacetime diagrams allows us to observe one of the most startling implications of special relativity prior to executing any math to prove its truth: *It is evident that events simultaneous in the unprimed frame are not simultaneous in the primed frame, and vice versa.* We can deduce, therefore, that simultaneity is completely dependent upon specification of the reference frame in which observations are made.

Let's work through an example that illustrates the loss of simultaneity with spacetime diagrams.

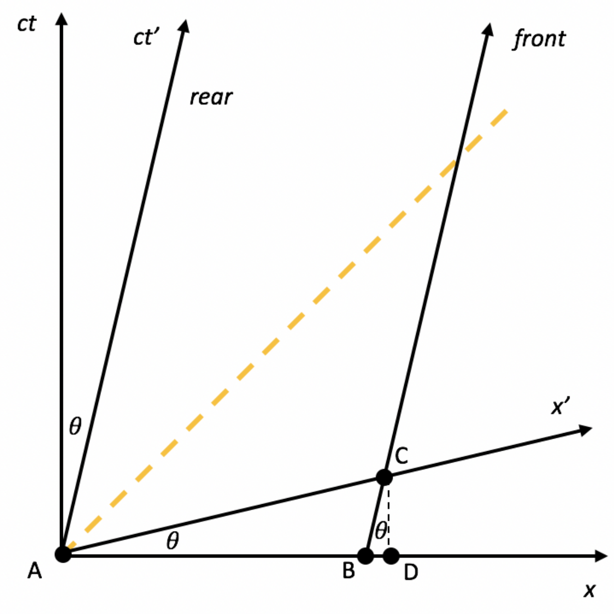
We first consider an observer in the unprimed frame that looks at a train of proper length L passing in the primed frame with a velocity v. At the front and back ends of the train are two clocks that are synchronized in the train's frame. We will show that these clocks are *not* synchronized at simultaneous times in the frame of the observer.

The situation in the unprimed frame of the observer is illustrated below.

Since $AC = 1$ meter in the primed frame, and since one unit along the x' axis has length $\sqrt{\frac{1+\beta^2}{1-\beta^2}}$ in the unprimed frame as determined by its distance to the origin, this radical also represents the length of AC.

$$AC = L\sqrt{\frac{1+\beta^2}{1-\beta^2}} \tag{1}$$

$$CD = AC\sin\theta \tag{2}$$



$$
\begin{aligned}
CB &= \frac{CD}{\cos\theta} \\
&= \frac{AC\sin\theta}{\cos\theta} \\
&= L\beta\sqrt{\frac{1+\beta^2}{1-\beta^2}} \\
&= L\frac{v}{c}\sqrt{\frac{1+\beta^2}{1-\beta^2}}
\end{aligned}
\tag{3}
$$

Since ct' axis is stretched by the same factor, $\frac{1+\beta^2}{1-\beta^2}$, as the x' axis, our expression for CB must be reduced by this same factor in order to yield the true length CB in the unprimed frame.

$$
\begin{aligned}
ct' &= CB \\
&= \frac{L\frac{v}{c}\sqrt{\frac{1+\beta^2}{1-\beta^2}}}{\sqrt{\frac{1+\beta^2}{1-\beta^2}}} \\
&= L\frac{v}{c}
\end{aligned}
\tag{4}
$$

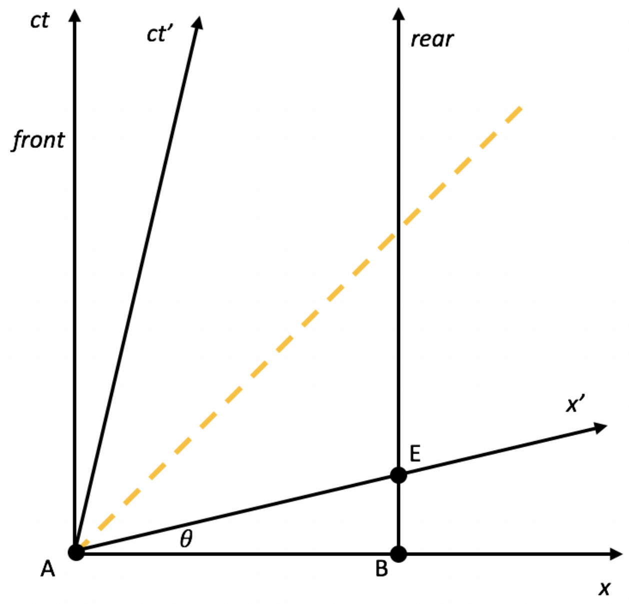
This result represents the delay that the observer reads on the clock at the front of the train with respect to the clock at the rear when the two are measured simultaneously in the unprimed frame.

$$
t' = L\frac{v}{c} \tag{5}
$$

The unprimed observer sees that when measured simultaneously, the clock at the rear of the train is ahead of the clock at the front by $L\frac{v}{c^2}$. In the primed frame, though, these clocks are synchronized, demonstrating that simultaneity depends completely upon specification of one's reference frame.

Let's now look at the case when the observer is stationed in the primed frame and sees the train of proper length L pass him in the unprimed frame. The situation is illustrated below.

Length AB represents L, the proper length of the train, so BE has a length $L\tan\theta = L\frac{v}{c}$. Since BE measures the positive time difference between the clock at the rear versus that at the front:



$$ct = L\frac{v}{c} \tag{6}$$

$$t = L\frac{v}{c^2} \tag{7}$$

The observer in the primed frame sees the difference between rear and front clocks as $L\frac{v}{c^2}$ when evaluated simultaneously in the primed frame.

Length Contraction

In the previous example, the idea of the train having a 'proper length' and a different length from the perspective of an unprimed observer was briefly touched upon. Let's explore these length disparities more thoroughly with the example of measuring a meter stick in stationary and moving reference frames.

First, the meter stick rests in the unprimed frame with its left end at the origin of the spacetime diagram. Fundamentally, measuring the length of the stick requires taking the difference between the spatial coordinates of each end simultaneously. The meter stick, therefore, lies along the x axis with a length AB. In the unprimed reference frame, this length is one meter. But the distance AM, which represents the meter stick's length in the primed frame traveling past with velocity v, is clearly different than AB.

We can use a Lorentz transformation to obtain the coordinate along the x' axis for M in terms of x.

$$x' = \frac{x - vt}{\sqrt{1 - v^2}}$$

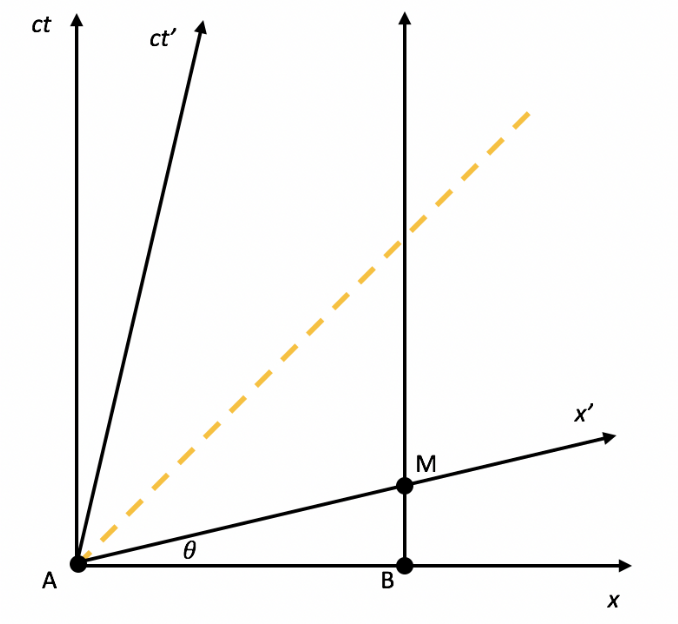
In this case, $x' = M$ and $x = B$. The x' axis is given by the equation $t = vx$. Substitute in for t to yield

$$x' = \frac{x - v^2t}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{8}$$

$$x' = x\sqrt{1 - \beta^2} \tag{9}$$

If this is the scale factor for the spatial axes, it also represents the scale factor for the meter stick's length.

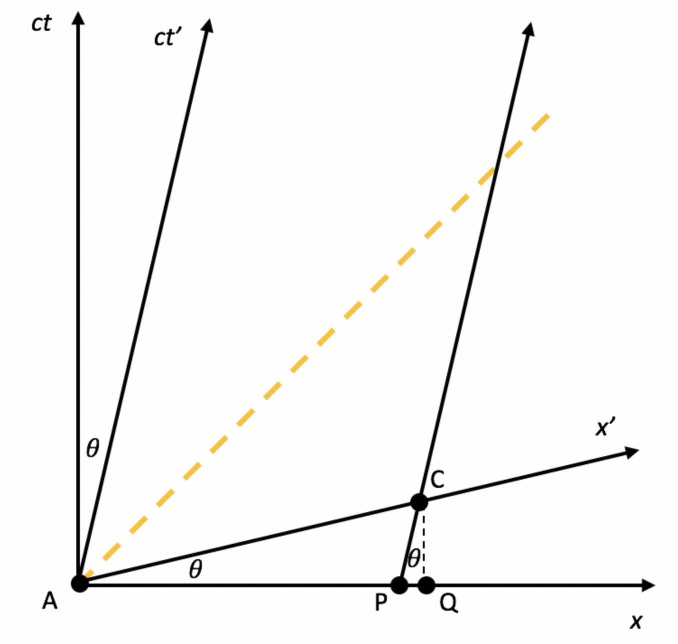
$$L' = L\sqrt{1 - \beta^2} \tag{10}$$



$$AM = AB\sqrt{1-\beta^2} \tag{11}$$

The meter stick resting in the unprimed frame is observed to be shorter by a factor of $\sqrt{1-\beta^2}$ by an observer passing in the primed frame.

If we repeat the process but allow the stick to be at rest in the primed frame, then the spacetime diagram from the perspective of an observer in the unprimed frame appears as follows.



The ends of the meter stick are represented by the ascending parallel diagonal lines. The left end, A, rests at the origin and the stick has length $L' = 1$ meter in the primed frame, shown by AC. We seek AP, the length of the meter stick as observed in the unprimed frame. Like that which separates the x and x' axes, the angle between the ct and ct' axes is θ, meaning angle PCQ must also be equivalent.

Recalling that one unit along the x' axis has length $\sqrt{\frac{1+\beta^2}{1-\beta^2}}$ in the unprimed frame as determined by its distance to the origin, this radical also represents the length of AC.

$$\tan\theta = \beta \tag{12}$$

$$CQ = (AC)\sin\theta \tag{13}$$

$$\begin{aligned} PQ &= (CQ)\tan\theta \\ &= (AC)\sin\theta\tan\theta \end{aligned} \tag{14}$$

$$\begin{aligned} AP &= AQ - PQ \\ &= (AC)\cos\theta - (AC)\sin\theta\tan\theta \\ &= \sqrt{\frac{1+\beta^2}{1-\beta^2}}\frac{1}{\sqrt{1+\beta^2}}(1-\beta^2) \\ &= \sqrt{1-\beta^2} \end{aligned} \tag{15}$$

$$L = L'\sqrt{1-\beta^2} \tag{16}$$

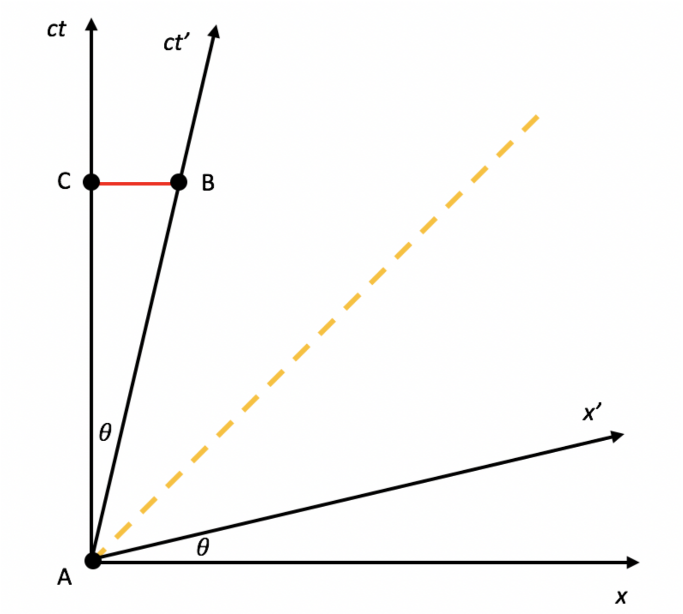
We obtain the same factor of length contraction as before. In the unprimed frame, an observer sees that the meter stick has shrunk by a factor of $\sqrt{1-\beta^2}$ as it flies by her in the primed frame.

Note that all length contraction occurs *in the direction of motion.* No component of length contraction is in the perpendicular direction; if an observer measures a three-foot-tall vase instead of a meter stick, she would find the vase contracted only along its horizontal axis parallel to the direction of its motion. Its original three-foot height would be preserved.

Time Dilation

The final implication of special relativity that can be easily illustrated through spacetime diagrams is time dilation. We'll follow an example similar to the case of length contraction.

Let's first examine the case where an observer in the unprimed frame sees a clock, stationary in the primed frame, fly past with velocity v. One second of time passes in the interval between A and B, so the length of AB, as previously shown, equals $\sqrt{\frac{1+\beta^2}{1-\beta^2}}$.



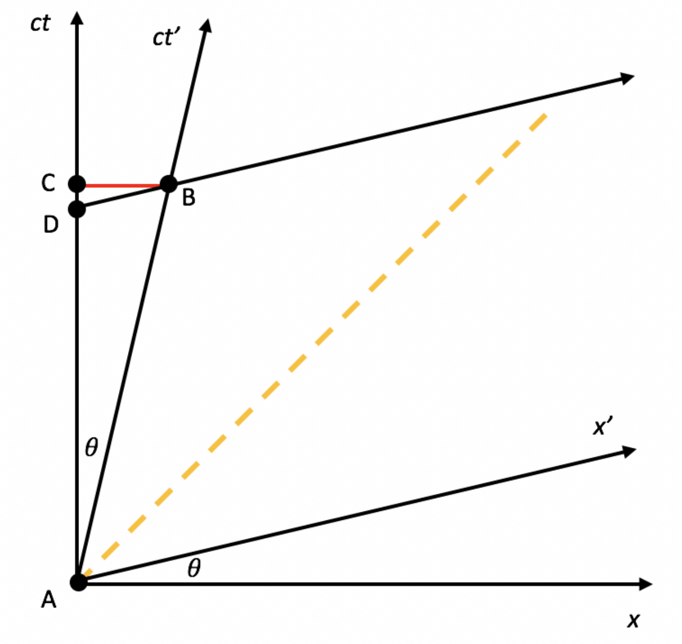
In the unprimed frame, B and C are two simultaneous events. The interval AC will determine how many seconds pass according to the unprimed observer for every one second that passes on the clock.

$$
\begin{aligned}
AC &= (AB)\cos\theta \\
&= \sqrt{\frac{1+\beta^2}{1-\beta^2}}\frac{1}{\sqrt{1+\beta^2}} \\
&= \frac{1}{\sqrt{1-\beta^2}}
\end{aligned} \tag{17}
$$

$$
t' = \frac{t}{\sqrt{1+\beta^2}} \tag{18}
$$

More time elapses for the unprimed observer than for the primed clock within the same interval.

We can also consider the case where the observer in the primed frame sees a clock at rest in the unprimed frame flying past at speed v. Whereas previously, the clock's worldline was given by the ct' axis, it now lies along the ct axis. B and D are simultaneous in the observer's primed frame, while C and B are simultaneous in the unprimed frame of the clock.



The angle CBD is also equal to θ. If one second on the clock is measured by the interval AB, then the length of this segment is once more $\sqrt{\frac{1+\beta^2}{1-\beta^2}}$. The interval AC determines how much time passes for the observer in the primed frame for every one second that has passed on the unprimed clock. Let's say AC has length L.

$$
\tan\theta = \beta \tag{19}
$$

$$
CD = L\sin\theta \tag{20}
$$

$$
\begin{aligned}
BD &= (CD)\tan\theta \\
&= L\sin\theta\tan\theta \\
&= AB \\
&= AD - BD \\
&= L\cos\theta - L\sin\theta\tan\theta \\
&= (L\cos\theta)(1-\tan^2\theta) \\
&= L\frac{1-\beta^2}{\sqrt{1+\beta^2}}
\end{aligned} \tag{21}
$$

where

$$L = (AB)\frac{1-\beta^2}{\sqrt{1+\beta^2}} \tag{22}$$

AB corresponds to one second in unprimed frame and has length $\sqrt{\frac{1+\beta^2}{1-\beta^2}}$, so

$$L = \frac{\sqrt{1+\beta^2}}{1-\beta^2} \tag{23}$$

and

$$\begin{aligned} AC &= \frac{\sqrt{\frac{1+\beta^2}{1-\beta^2}}}{L} \\ &= \sqrt{\frac{1+\beta^2}{1-\beta^2}}\frac{1-\beta^2}{\sqrt{1+\beta^2}} \\ &= \frac{1}{\sqrt{1-\beta^2}} \end{aligned} \tag{24}$$

$$t' = \frac{t}{\sqrt{1+\beta^2}} \tag{25}$$

We obtain the same time dilation result as before: the primed observer passing the unprimed clock would experience more time elapsing within the same interval. We have now provided three examples of how spacetime diagrams can be used to help make sense of and visualize the perplexing effects of special relativity.

Sources

Each mathematical proof follows the work in David Morin's *Introduction to Classical Mechanics*:

Morin, D. (2007). Relativity (Kinematics). In *Introduction to Classical Mechanics* (pp. 501-583). Cambridge: Cambridge University Press.