The Double Pendulum

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Abstract

This paper offers an overview of the most useful methods for exploring the double pendulum – a simple physical system with a strong sensitivity to initial conditions that displays erratic, dynamic behavior. The energy of the system will be determined and used to derive its equations of motion. With these equations in hand, we will adjust the parameters of the system in order to find mathematical solutions and physical insights for various conditions and associated behaviors, including the equations of motion and frequency of oscillation when the system is perturbed by a small angle. The small-angle approximation will be followed by investigation of a number of special cases when ratios between the two masses and string lengths are pre-specified. We will close with a discussion of the double pendulum's chaotic tendencies and the effects of varied initial conditions.

Introduction

The double pendulum is composed of a pendulum with another pendulum attached to its end. Thus, its critical components for the purpose of mathematical and physical exploration are its two masses, m_1 and m_2, its two string lengths, l_1 and l_2, and the angles formed between these strings and the vertical, denoted as θ_1 and θ_2.

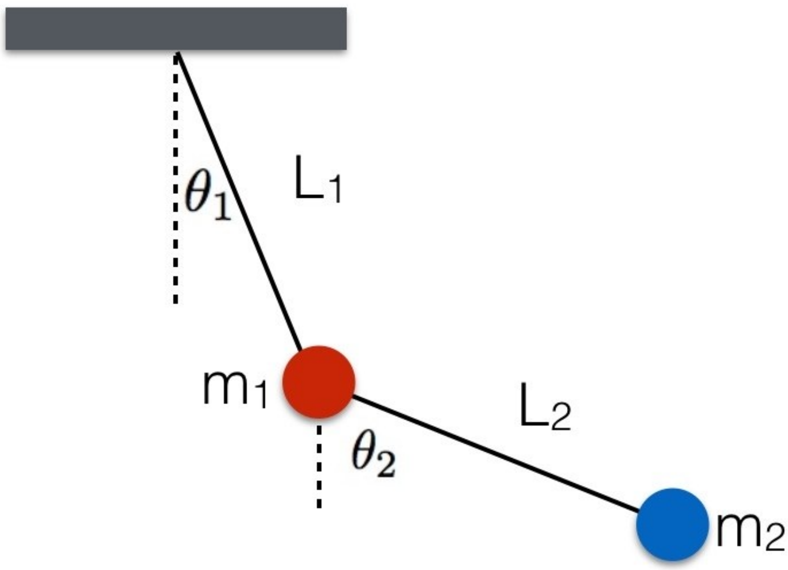

Figure 1: The double pendulum consists of a second pendulum attached to the bob of a first. (source: Wired.com)

While both simple and double pendulums can be approximated to exhibit simple harmonic oscillation at small angles of perturbation, the behavior of the double pendulum is far more difficult to categorize and predict. The presence of multiple masses, string lengths and angles permit more conditions to be independently adjusted. Fixing various parameters leads to a host of interesting dynamical results, many of which cannot arise in a simple pendulum, and provides an intuitive sense of why initial conditions are so critical in determining the system's behavior. Such dependence on these conditions makes the double pendulum system chaotic, meaning approximations are not enough to predict future behavior and even slight differences in initial parameters will eventually lead to vast disparities in the motions of two systems. We will later show that tweaking initial conditions indeed results in these types of disparities in the numerical outcome.

The Lagrangian Model

A further consequence of the double pendulum's heightened complexity and sensitivity is the difficulty one faces in trying to analyze it from a Newtonian point of view. For this system and those with similarly rich dynamical motion, attempting to determine equations of motion through Newtonian force diagrams with various angular components of multiple tensions and gravitational effects can quickly become exasperating. A far simpler method to obtain these same equations is the Lagrangian approach, which merely requires computation of the system's kinetic and potential energies.

The Lagrangian equation is given by

$$L = T - U \tag{1}$$

where T is the kinetic energy and U the potential. After writing down the Lagrangian, the system's equations of motion can be obtained through the generalized Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{2}$$

The Euler-Lagrange equation requires partial derivatives of the Lagrangian with respect to both a variable and its derivative. It is important to note in the manipulation of the Lagrangian and the Euler-Lagrange equation that the derivative $\dot{q}$ is treated as its own variable: we can – and must – differentiate the Lagrangian with respect to $\dot{q}$ while holding q fixed. After determining the two partial derivatives of L, we must remember that the third step in solving the Euler-Lagrange equation requires a total derivative with respect to t of the partial with respect to $\dot{q}$ that we have obtained. This means that each element of the partial that depends on time must be accounted for, likely requiring at least one iteration of the Chain Rule.

We must also remember that an Euler-Lagrange equation must be derived for each pertinent, time-dependent variable. For a system whose Lagrangian depends on $x(t)$, $y(t)$, $z(t)$, for example, we would need to obtain an Euler-Lagrange equation for all three variables. In the case of the double pendulum, as we shall soon see, we will need two equations corresponding to θ_1 and θ_2.

In order to go about solving the double pendulum using the Lagrangian approach, we must start by defining the positions of each mass in Cartesian coordinates. From the geometry of the given figure, those positions are

$$x_1 = l_1 \sin\theta_1 \qquad y_1 = -l_1 \cos\theta_1 \tag{3}$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \qquad y_2 = -l_1 \sin\theta_1 - l_2 \cos\theta_2 \tag{4}$$

$$\tag{5}$$

We can differentiate each of the positions in order to obtain the $\dot{x}$ and $\dot{y}$ values that will be essential to writing the system's kinetic energy.

$$\dot{x}_1 = l_1\dot{\theta}_1 \cos\theta_1 \qquad \dot{y}_1 = l_1\dot{\theta}_1 \sin\theta_1 \tag{6}$$

$$\dot{x}_2 = l_1\dot{\theta}_1 \cos\theta_1 \dot{x}_1 + l_2\dot{\theta}_2 \cos\theta_2 \qquad \dot{y}_2 = l_1\dot{\theta}_1 \sin\theta_1 \dot{x}_1 + l_2\dot{\theta}_2 \sin\theta_2 \tag{7}$$

$$\tag{8}$$

From here, we can start constructing the expressions for T and U. The general form for each is

$$T = \frac{1}{2} m_n \dot{q}_i^2 \qquad U = -m_n g y_n \tag{9}$$

As we know, the Lagrangian is the difference between kinetic and potential energies. Using the expressions for the velocities and positions of each mass in both x and y, the Lagrangian can be written as

$$\begin{aligned}
L &= \frac{1}{2}m_1(\dot{x}_1 + \dot{y}_1)^2 + \frac{1}{2}m_2(\dot{x}_1 + \dot{y}_1)^2 + m_1 g y_1 + m_2 g y_2 \\
&= \frac{1}{2}m_1(l_1^2\dot{\theta}_1^2) + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos[\theta_1 - \theta_2]) + m_1 g l_1 \cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)
\end{aligned} \tag{10}$$

From here, we can write down the appropriate partials for each angle θ_1 and θ_2, using these to construct the two Euler-Lagrange equations. Remember to take full advantage of trigonometric identities in order to simplify these equations.

$$0 = \ddot{\theta}_1 l_1^2(m_1 + m_2) + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \tag{11}$$

$$0 = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 \tag{12}$$

These are the equations of motion when the two angles have been varied. They're somewhat disorderly now, but as we shall see, they will become much more streamlined after we've made approximations for small angles.

Small Oscillations

We discussed earlier how both simple and double pendulums can be modeled as simple harmonic oscillators when angles of perturbation are restricted in size. For sine and cosine, the Taylor Series expansions about the point $x = 0$ are

$$\begin{aligned}
\sin\theta &\approx \theta \\
\cos\theta &\approx 1 + \frac{\theta^2}{2!}
\end{aligned}$$

Expanding the condensed $\cos(\theta_1 - \theta_2)$ and $\sin(\theta_1 - \theta_2)$ terms, we can make the appropriate substitutions of these approximations into Eqs. (11) and (12) and keep only linear-order terms, yielding

$$0 = \ddot{\theta}_1 l_1^2(m_1 + m_2) + g\theta_1(m_1 + m_2) + m_2 l_2 \ddot{\theta}_2(1 + \theta_1\theta_2) +,_2 l_1 l_2 \dot{\theta}_2^2(\theta_2 - \theta_1) \tag{13}$$

$$0 = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1(1 + \theta_1\theta_2) - m_2 l_1 \dot{\theta}_1^2(\theta_2 - \theta_1) + m_2 g\theta_2 \tag{14}$$

These equations can be further simplified after a few observations. We can eliminate all terms containing $\dot{\theta}^2$, since assuming small angles of oscillation entails assuming small angular velocities. Squares of already-small velocities are negligibly small, so the expressions with $\dot{\theta}^2$ terms can be eliminated.

Additionally, we have terms of the form $\theta_1\theta_2$. If both angles are very small, however, then multiplying them by each other results in a value that is negligibly small, for reasons similar to those that allowed us to drop higher-order terms in the Taylor Series for sine and cosine.

Implementing these changes and simplifying, we find that the equations of motion for small angle approximations are

$$0 = \ddot{\theta}_1 l_1^2(m_1 + m_2) + g\theta_1(m_1 + m_2) + m_2 l_2 \ddot{\theta}_2 \tag{15}$$

$$0 = l_2^2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 + g\theta_2 \tag{16}$$

Specific Cases

Armed with these equations of motion, we can adjust their parameters to examine the frequencies of oscillation for special cases. Let's look at the case when $l_1 = l_2 = l$ and $m_1 = m_2 = m$. As we shall see, the previously jumbled equations of motion will simplify to a concise expression for the system's oscillating frequencies. From here, we'll be able to solve for the normal modes of the system, completely characterizing its motion for small oscillations under these special conditions.

We start by equating the string lengths and masses to render more useful equations of motion.

$$0 = m(2\ddot{\theta}_1 l + 2g\theta_1 + l\ddot{\theta}_2) \tag{17}$$

$$0 = l\ddot{\theta}_2 + l\ddot{\theta}_1 + g\theta_2 \tag{18}$$

We must guess the solutions to these differential equations, but we know that they will likely take the form of exponentials, whose derivatives yield the same root expression multiplied by a different coefficient.

Let's make the logical guess for the time-dependent solution for each angle.

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} = \begin{pmatrix}A\\ B\end{pmatrix} e^{i\omega t} \tag{19}$$

Using these expressions for $\theta_1(t)$ and $\theta_2(t)$, we can substitute the second derivatives into the appropriate positions within the equations of motion. The second derivatives can be written as

$$\begin{pmatrix}\ddot{\theta}_1(t)\\ \ddot{\theta}_2(t)\end{pmatrix} = \begin{pmatrix}-A\omega^2\\ -B\omega^2\end{pmatrix} e^{i\omega t} \tag{20}$$

rendering new equations of motion

$$0 = me^{i\omega t}(-2lA\omega^2 + 2gA - lB\omega^2) \tag{21}$$

$$0 = e^{i\omega t}(lA\omega^2 - lB\omega^2 + gB) \tag{22}$$

The exponentials can be eliminated from each expression since they will never equal 0, and the mass can be factored out of the first expression. We then substitute α for ω^2, as is customary in the execution of this method, and we form the matrix representing this system of equations.

$$\begin{pmatrix}2l\alpha - 2g & l\alpha\\ -l\alpha & g - l\alpha\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} = 0 \tag{23}$$

The determinant of this matrix must equal 0 in order to yield an interesting solution. Taking this determinant, we find an equation that simplifies to

$$l^2\alpha^2 - 4gl\alpha + 2g^2 = 0$$

Solving this equation for α with the quadratic formula yields the frequencies of oscillation, $\omega_\pm$, that we seek.

$$\omega_\pm = \sqrt{2 \pm \sqrt{2}}\sqrt{\frac{g}{l}}$$

Taking the positive and negative solutions and plugging each back into the matrix for α, the normal modes, as confirmed in Morin's *Introduction to Classical Mechanics*, can be derived.

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} = \begin{pmatrix}-1\\ \sqrt{2}\end{pmatrix}\cos(\omega_+ t + \delta) + \begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}\cos(\omega_- t + \phi) \tag{24}$$

Shown below are graphs of each normal mode in order, with $g = 9.8$, $l = 1$, and $\delta = \phi = 0$.

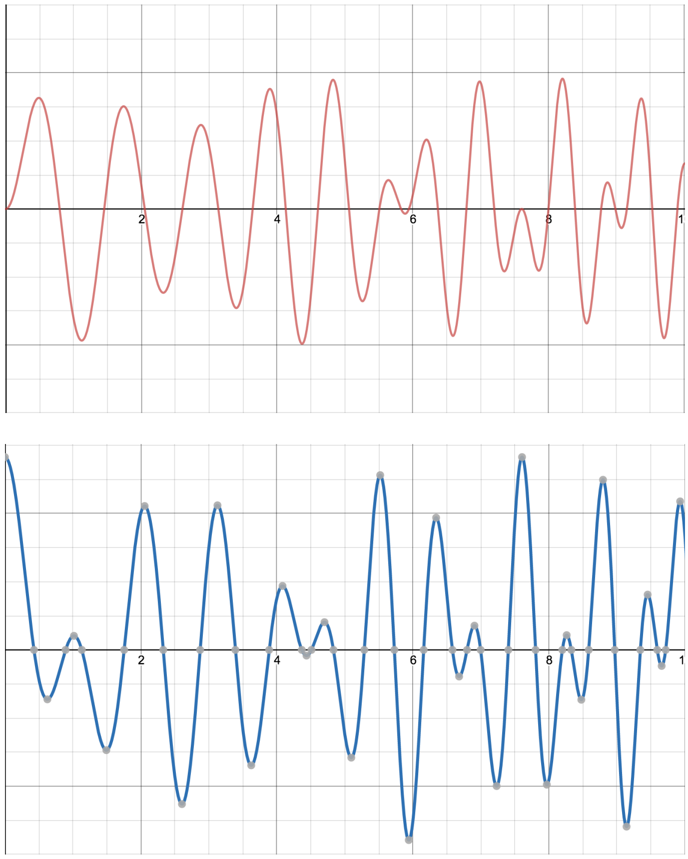

Figure 2: The graphs for the normal modes. $\theta_1(t)$ is in red, with $\theta_2(t)$ in blue.

Limits

Let's examine a couple of other special cases that specifically concern limits.

When $m_1 >> m_2$

Referring back to the double pendulum diagram, this limit involves a top mass much greater than the bottom mass. We can make a guess about the behavior of this limited system and the nature of the associated frequencies and normal modes based on intuition. If $m_1 >> m_2$, then m_1 will essentially be fixed in place, standing still and leaving m_2 to swing freely as a simple pendulum with length l.

To go about solving for the frequencies of this system, we must approximate the relationship between m_1 and m_2. This process resembles that shown in chapter six of Morin. The condition $m_1 >> m_2$ can be written as $1 >> \frac{m_2}{m_1}$. We can express the fraction as κ.

$$1 >> \kappa$$

It can be shown that the frequencies of oscillation at small angles for equal string lengths but unequal masses are given by

$$\omega_{\pm} = \sqrt{\frac{g}{l}}\sqrt{\frac{m_1 + m_2 \pm \sqrt{m_1 m_2 + m_2^2}}{m_1}} \tag{25}$$

This is the expression into which we must insert the inequality, but first it needs to be transformed into a more manageable form. It's easiest to approximate an exponentiated expression by manipulating it into the format

$$(1 + x)^n \approx 1 + nx$$

In our case, we can simply carry out the fraction inside the radical to render an expression quite close to this format

$$\begin{aligned} \omega_{\pm} &= \sqrt{\frac{g}{l}}\sqrt{1+\kappa \pm \sqrt{\frac{m_1 m_2 + m_2^2}{m_1^2}}} \\ &= \sqrt{\frac{g}{l}}\sqrt{1+\kappa \pm \sqrt{\kappa+\kappa^2}} \end{aligned} \tag{26}$$

Following the pattern of the approximation, the frequency becomes

$$\sqrt{\frac{g}{l}}(1+\frac{\kappa}{2} \pm \frac{1}{2}\sqrt{\kappa+\kappa^2}) \tag{27}$$

We know that κ is very small, so it can be removed from the expression. Likewise, κ^2 is trivially small, leaving us with a solution of frequencies

$$\omega_{\pm} = \sqrt{\frac{g}{l}}(1 \pm \frac{\sqrt{\kappa}}{2}) \tag{28}$$

with normal modes

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} -\sqrt{\kappa} \\ 1 \end{pmatrix} \cos(\omega_+ t + \delta) + \begin{pmatrix} \sqrt{\kappa} \\ 1 \end{pmatrix} \cos(\omega_- t + \phi) \tag{29}$$

As expected, the heavy upper mass is nearly motionless while the lighter bottom oscillates as a simple pendulum.

When $l_2 >> l_1$

In this case, the length of the bottom string is much greater than that of the top string. The double pendulum features one mass essentially on the junction point of the system. The other mass hangs as a simple pendulum of length l_2.

We can approach this limit the same way we approached the last: by representing $l_2 >> l_1$ as $1 >> \frac{l_1}{l_2}$.

$$1 >> \kappa$$

It can be shown that the frequencies of small oscillations for equal masses but unequal string lengths are

$$\omega_{\pm} = \sqrt{\frac{l_1 + l_2 \pm \sqrt{l_1^2 + l_2^2}}{l_1 l_2}}\sqrt{g} \tag{30}$$

Using the same approximation method employed previously, the frequencies to non-trivial order of κ are

$$\omega_{\pm} = \sqrt{\frac{2g}{l_1}} \qquad\qquad \omega_{-} = \sqrt{\frac{g}{l_2}} \tag{31}$$

The normal modes, as confirmed in Morin, are

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} \cos(\omega_+ t + \delta) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(\omega_- t + \phi) \tag{32}$$

These normal modes indicate that the system displays the two expected behaviors. In one case, the top mass oscillates at high frequency, while in the second, the bottom mass functions as a simple pendulum of length l_2.

Chaos and Numerical Solutions

As mentioned previously, the double pendulum exhibits rich dynamical motion and extreme sensitivity to its parameters. These behaviors are characteristic of chaotic systems, or those subject to nonlinear oscillation. Linearity of oscillation is implicit within a system's equations of motion: if the oscillating variable or its derivative is present in powers higher than linear, the system exhibits nonlinear behavior.

Indeed, in examining the full equations of motion for the double pendulum, Eqs. 13 and 14 prior to approximation for small angles, we find order-two expressions of $\ddot{\theta}_1$ and $\ddot{\theta}_2$.

$$
\begin{aligned}
0 &= \ddot{\theta}_1 l_1^2 (m_1 + m_2) + g\theta_1 (m_1 + m_2) + m_2 l_2 \ddot{\theta}_2 (1 + \theta_1\theta_2) + l_2 l_1 l_2 \dot{\theta}_2^2 (\theta_2 - \theta_1) \\
0 &= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 (1 + \theta_1\theta_2) - m_2 l_1 \dot{\theta}_1^2 (\theta_2 - \theta_1) + m_2 g\theta_2
\end{aligned}
$$

Chaos is defined as the motion of a system whose evolution in time sensitively depends on initial conditions. Alternatively, it can be thought of as a system's precarious dependence on its previous state. Although we undoubtedly know the equations governing the motion of such a system, measurements of its state at a particular point in time may not allow us to accurately predict future states even in the very near future.

Examples of chaotic behavior abound in the natural world: epidemics, weather patterns, and changing animal populations are just a few systems responsive to internal and initial conditions that are inherently unpredictable in the long term.

We can get a taste of the implications of chaotic behavior in the double pendulum by modeling its motion with Mathematica and tweaking initial parameters, namely the initial angles $\theta_1(0)$ and $\theta_2(0)$. For simplicity, initial angular velocities $\dot{\theta_1(0)}$ and $\dot{\theta_2(0)}$ will be fixed at 0, but these are also viable initial parameters to vary and observe in their own right.

For the first part of this exploration, the initial angles are equal and have been set to two different values. For each value, we will let the system evolve for seven distinct timeframes and show the motion at each timeframe across both sets of initial conditions. The two sets of initial angles from the vertical chosen for both $\theta_1(0)$ and $\theta_2(0)$ are $\frac{\pi}{2}$ and $\frac{7\pi}{12}$, two arbitrary values with a disparity of $\frac{\pi}{12}$ between them. The timeframes at which paths of motion have been captured are 10, 20, 30, 40, 50, 60, and 90 seconds.

Table 1: Chaotic motion for initial angle disparity of $\frac{\pi}{12}$

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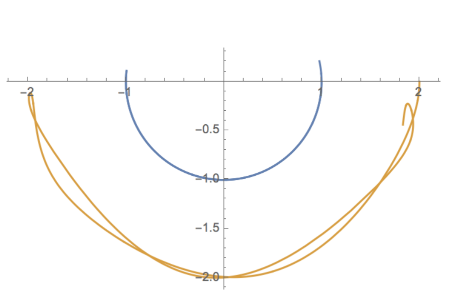
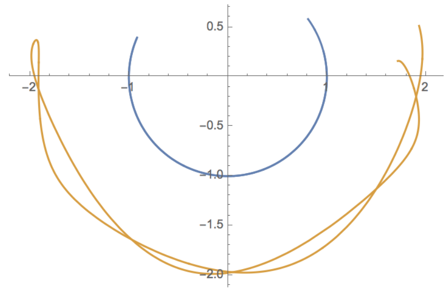
<tr><td>10</td><td>
</td><td>
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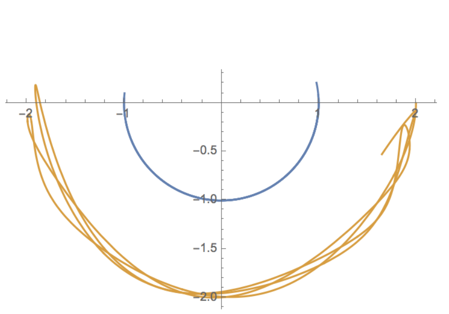
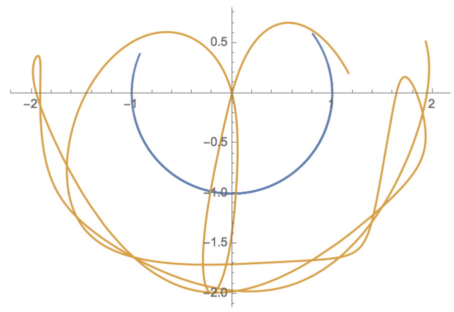
<tr><td>20</td><td>
</td><td>
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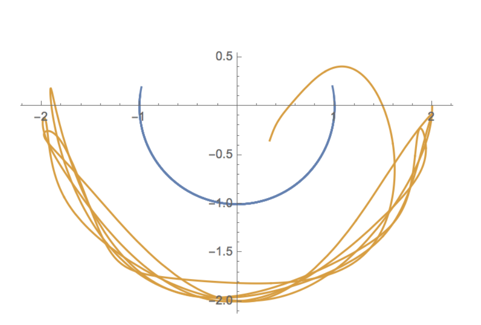
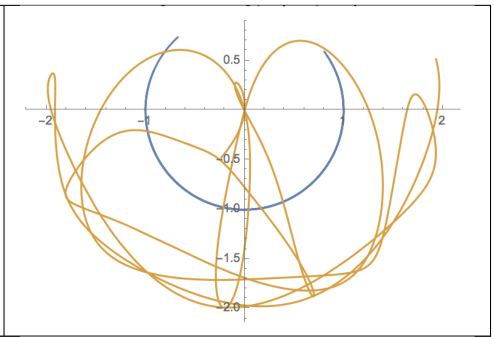
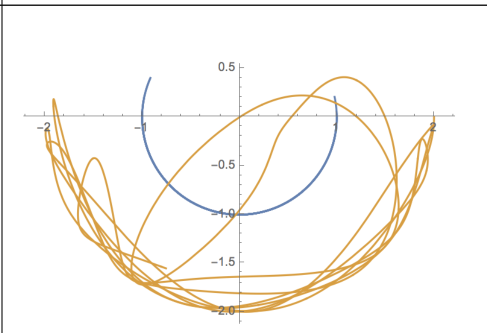
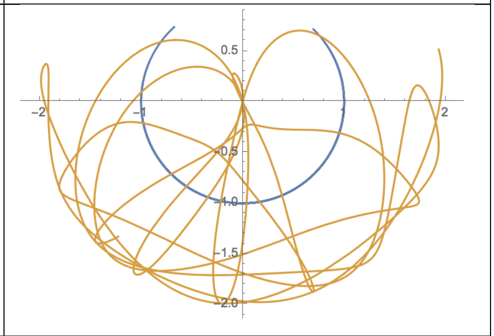
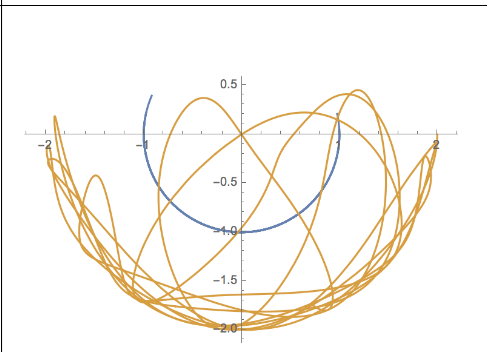
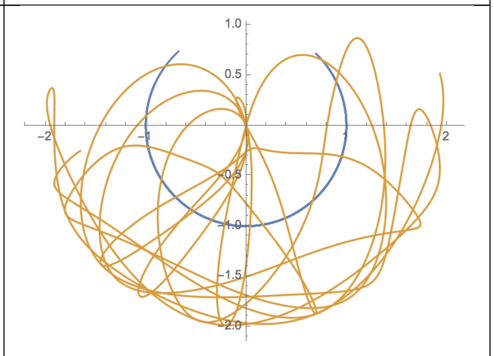
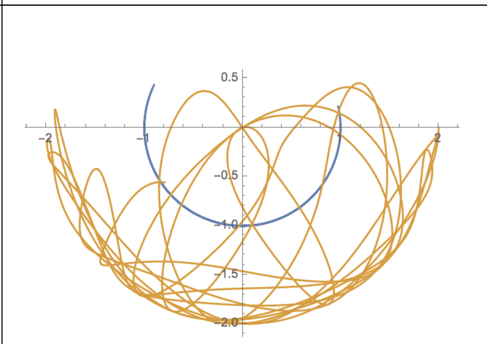
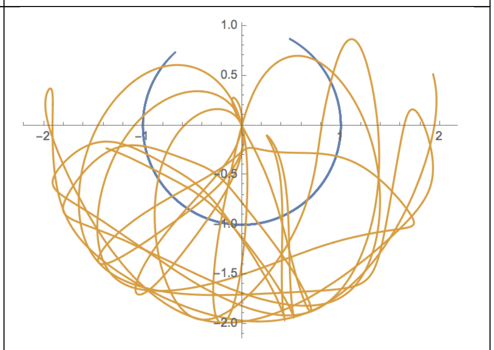
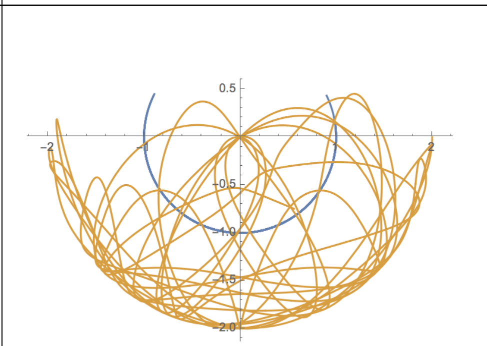
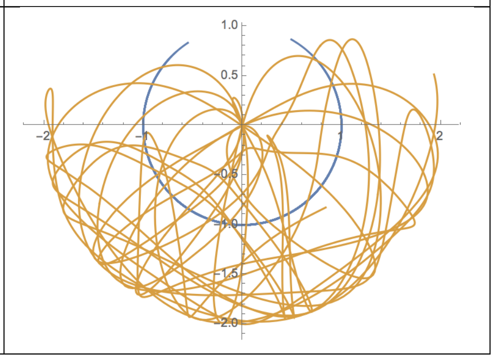
30		
40		
50		
60		
90		

When presented with two different initial conditions the system clearly exhibits discrepancies, especially the bottom mass, which swings wildly and traces the chaotic pattern of each figure. The top mass, as we shall observe for most initial condition scenarios, traces a neat circle or partial circle despite the behavior of the bottom mass. Note that we observe a 'flip' of the bottom mass one timeframe earlier in the $\frac{7\pi}{2}$ diagrams than in the $\frac{\pi}{2}$.

Let's see if discrepancies persist with an even smaller difference between the two sets of initial conditions by repeating the same trial with $\frac{\pi}{2}$ and $\frac{\pi}{2.01} = \frac{100\pi}{201}$. These values are only separated by an interval of $\frac{\pi}{402} = 0.0078$.

Table 2: Chaotic motion for initial angle disparity of $\frac{\pi}{402}$

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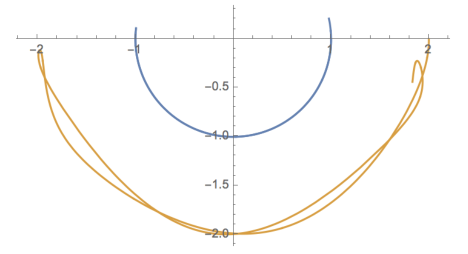
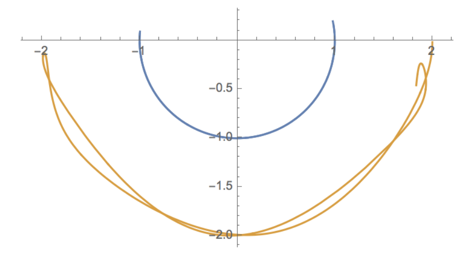
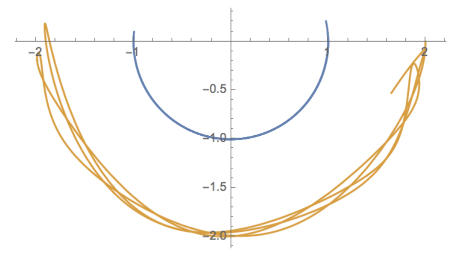
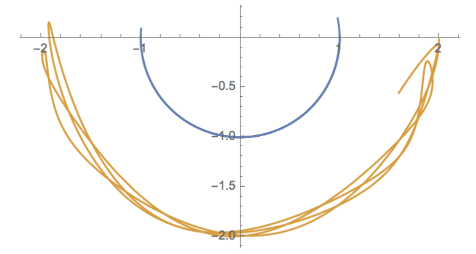
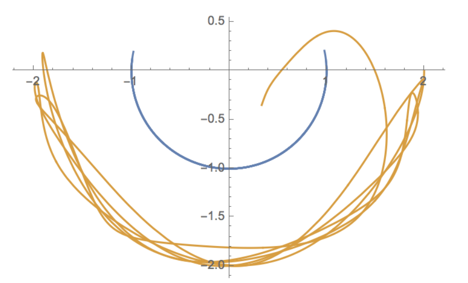
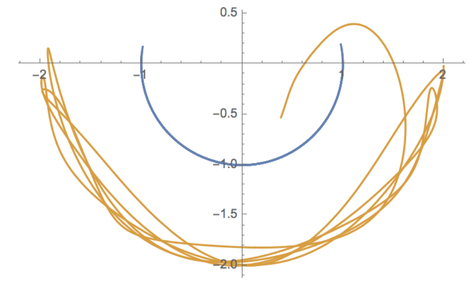
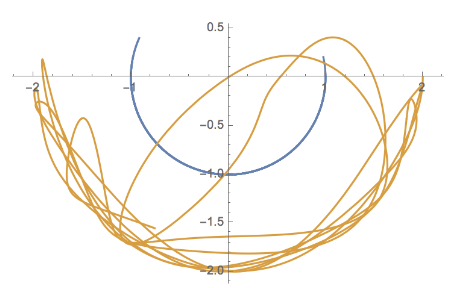
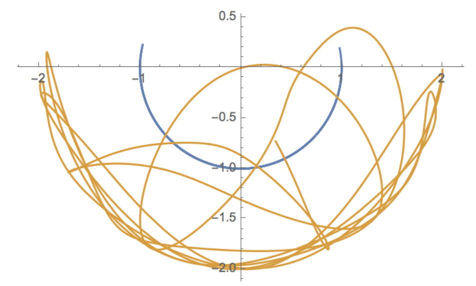
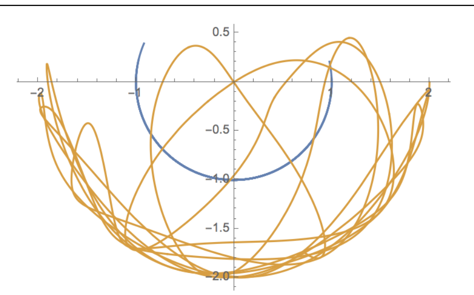
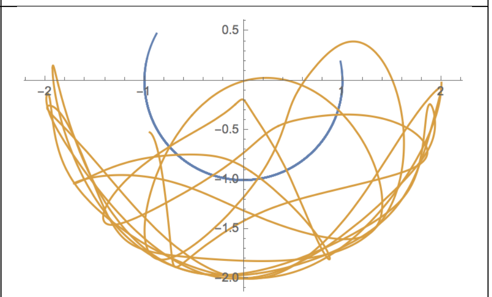
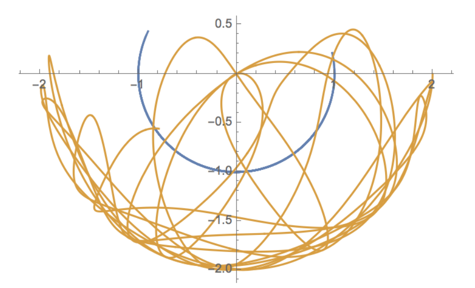
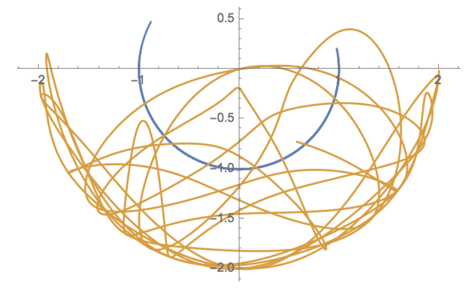
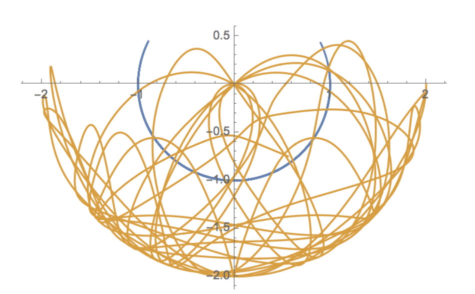
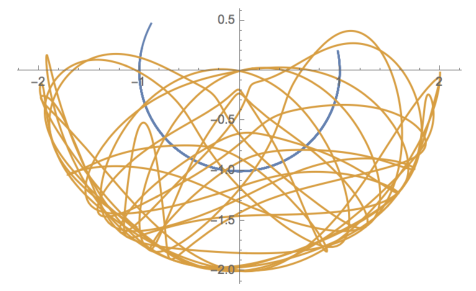
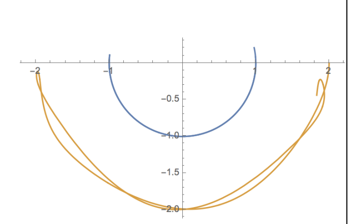
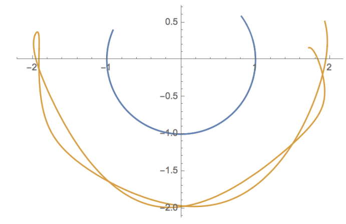
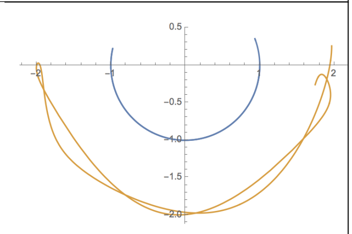
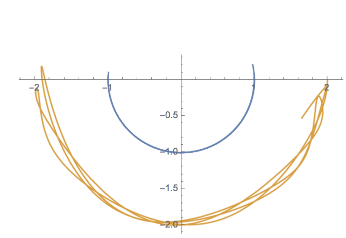
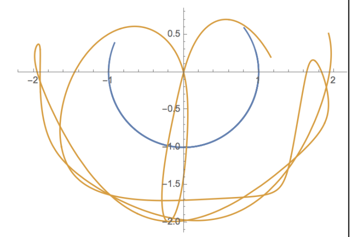
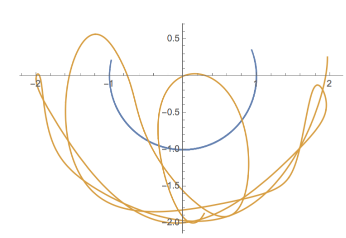
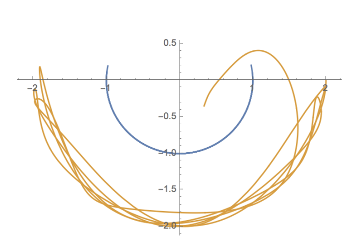
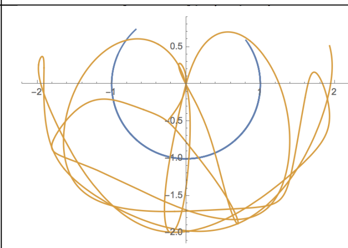
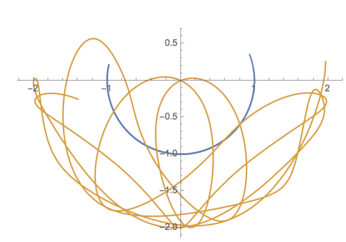
10		
20		
30		
40		
50		

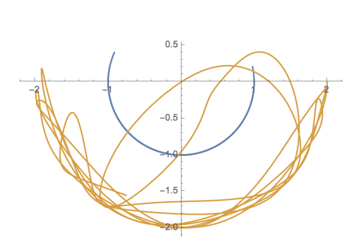
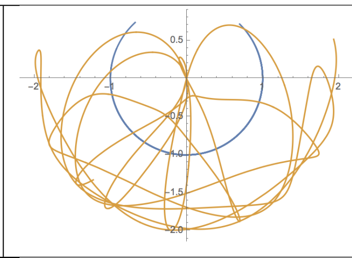
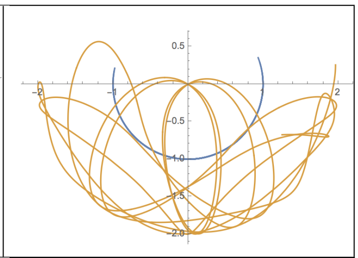
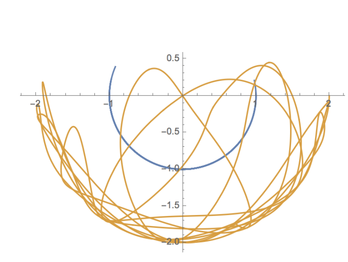
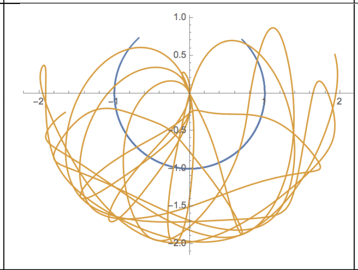
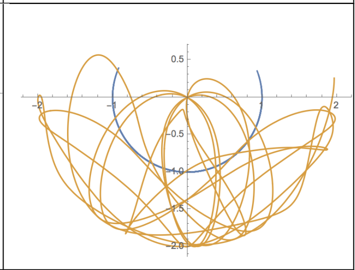
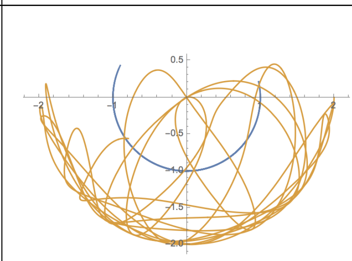
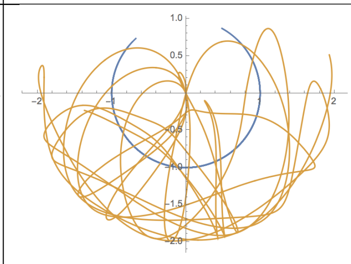
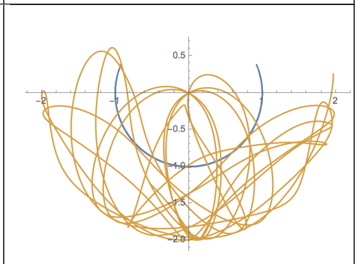
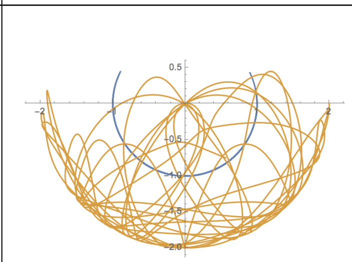
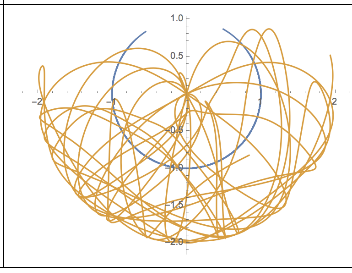
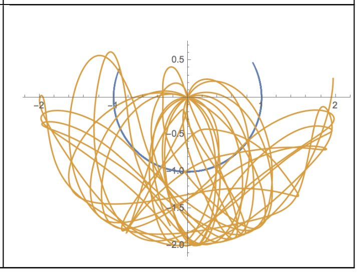
60		
90		

The differences in motion are not huge but are still present, and they become clear after only 40 seconds of elapsed time!

Up to this point, the two initial angles $\theta_1(0)$ and $\theta_2(0)$ have been equivalent. As a final step, let's vary one of these parameters, so that $\theta_1(0) = \frac{\pi}{2}$ and $\theta_2(0) = \frac{7\pi}{2}$. We'll compare the motion to our original parameters.

Table 3: Chaotic motion for cases when initial angles are equal and different

Time (s)	$\theta_1(0) = \theta_2(0) = \frac{\pi}{2}$	$\theta_1(0) = \theta_2(0) = \frac{7\pi}{2}$	$\theta_1(0) = \frac{\pi}{2}$ **and** $\theta_2(0) = \frac{7\pi}{2}$
10			
20			
30			

40			
50			
60			
90			

Chaotic tendencies continue to persist, and the motion with unequal initial angles is distinctly diverse from that exhibited when both angles are equal to either $\frac{\pi}{2}$ or $\frac{7\pi}{2}$. This type of modeling provides an easy method of verifying and visualizing the peculiar behavior of a system whose motion is as mesmerizing as it is unpredictable.

For more interaction with the double pendulum and the ability to alter its various parameters, click here.

Sources

Morin, D. (2007). The Lagrangian Method. *In Introduction to Classical Mechanics* (pp. 218–280). Cambridge: Cambridge University Press.

Thornton, S. T., & Marion, J. B. (2004). Nonlinear Oscillations and Chaos. In *Classical Dynamics of Particles and Systems* (5th ed., pp. 144–181). Belmont, CA: Thomson Brooks/Cole.